III.

ON SOME THEOREMS WHICH CONNECT TOGETHER CERTAIN LINE AND SURFACE INTEGRALS.

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In transforming from one set of curvilinear coördinates to another, some of the differential expressions which appear in problems in Hydrokinematics and Electrokinematics, I have found the theorems * stated below useful.

Theorem. — Let U be any function of the two polar coördinates, r and θ, which, with its first space derivatives, is finite, continuous,

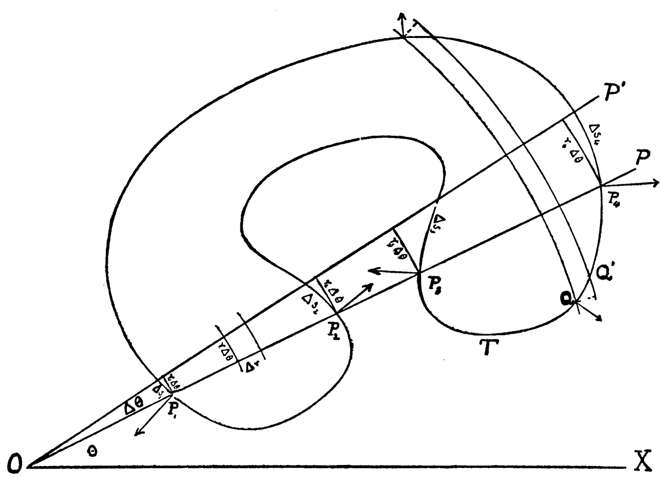

Fig. 1.

and single-valued throughout that part of the coördinate plane which is shut in by the closed curve T. Let δ be the angle between the radius vector, drawn from the origin to any point P on T, and the normal to T drawn from within outwards at P. Then, if T does not include

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the origin, the line integrals of $U \cos \delta$ and $U \sin \delta$, taken around T, are equal respectively to the surface integrals of $\frac{D_r(r \cdot U)}{r}$ and $\frac{D_\theta U}{r}$, taken over the area enclosed by T.

For the element of plane surface in polar coördinates, $r \Delta r \Delta \theta$ may be used. Let the radius vector OP, drawn so as to make the angle θ with the initial line OX, cut T $2n$ times at points $P_1, P_2, P_3, \ldots. P_{2n}$, distant respectively $r_1, r_2, r_3, \ldots. r_{2n}$ from O. Let the values of U at these points of intersection be $U_1, U_2, U_3, \ldots. U_{2n}$, respectively. Whenever the radius vector *cuts into* the closed contour, either $+\delta$ or $-\delta$ is an obtuse angle and $\cos \delta$ is negative; whenever the radius vector *emerges from* the space enclosed by the contour, either $+\delta$ or $-\delta$ is acute and $\cos \delta$ positive. The two neighboring radii vectores, OP and OP', which make with each other the angle $\Delta \theta$, include between them the arcs $\Delta s_1, \Delta s_2, \Delta s_3, \Delta s_4, \ldots. \Delta s_{2n}$, cut out of T, and the arcs $r_1 \Delta \theta, r_2 \Delta \theta, r_3 \Delta \theta, \ldots. r_{2n} \Delta \theta$, cut out of a set of circumferences drawn about O as centre, with radii $r_1, r_2, r_3, r_4, \ldots. r_{2n}$, respectively. It is evident that, if $\Delta \theta$ be made to approach zero as a limit,

$$+ \text{Limit} \frac{r_1 \cdot \Delta \theta}{\Delta s_1 \cdot \cos \delta_1} = - \text{Limit} \frac{r_2 \cdot \Delta \theta}{\Delta s_2 \cdot \cos \delta_2} = + \text{Limit} \frac{r_3 \cdot \Delta \theta}{\Delta s_3 \cdot \cos \delta_3}$$

$$= - \text{Limit} \frac{r_4 \cdot \Delta \theta}{\Delta s_4 \cdot \cos \delta_4} = + \text{Limit} \frac{r_{2n-1} \cdot \Delta \theta}{\Delta s_{2n-1} \cdot \cos \delta_{2n-1}} = - \text{Limit} \frac{r_{2n} \cdot \Delta \theta}{\Delta s_{2n} \cdot \cos \delta_{2n}}$$

$$= -1.$$

If the double integral be extended all over the space enclosed by T,

$$\int\!\!\int \frac{D_r(rU)}{r} \, r \, dr \, d\theta = \int d\theta \, [- r_1 U_1 + r_2 U_2 - r_3 U_3 + \ldots. + r_{2n} U_{2n}],$$

where the integration with respect to θ is to be extended over all values of the angle for which the corresponding radii vectores cut T. If now for $r_1 \Delta \theta$, $r_2 \Delta \theta$, $r_3 \Delta \theta$, etc., $-\cos \delta_1 \cdot ds_1$, $+\cos \delta_2 \cdot ds_2$, $-\cos \delta_3 \cdot ds_3 + \ldots. + \cos \delta_{2n} \, ds_{2n}$ be substituted respectively, we have

$$\int\!\!\int \frac{D_r(rU)}{r} \, r \, dr \, d\theta = \int [U_1 \cos \delta_1 \, ds_1 + U_2 \cos \delta_2 \, ds_2 + \ldots. U_{2n} \cos \delta_{2n}],$$

and this last integral is evidently equal to the line integral of $U \cos \delta$ taken all around T.

It is to be noticed that, if O were within T, each radius vector would cut T an odd number of times, and that a negative sign must stand before the line integral.

Since the limit of the ratio of Δr to any one, QQ', of the arcs cut out of T by two circumferences of radii r and $r + \Delta r$ respectively, drawn around O as a centre, is equal in absolute value to the sine of the angle which OQ makes with the external normal to T at Q, it is easy to prove the second part of the theorem by integrating $D_\theta U$ with regard to θ first, and then, after introducing proper limits, with regard to r.

This theorem may be regarded as a useful special case of the following

Theorem. — Let $\zeta = f_1(x, y)$ and $\eta = f_2(x, y)$ be two analytical functions of x and y such that the two families of curves $f_1(x, y) = c$, $f_2(x, y) = k$, are orthogonal. Let V be any function of x and y which, with its first space derivatives is finite, continuous, and single-valued within a closed curve T, drawn in the coördinate plane. Let h_1 and

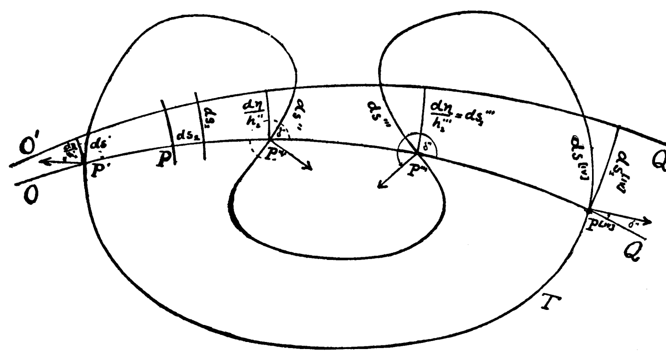

FIG. 2.

h_2 be the positive roots of the equations $h_1^2 = (D_x\,\zeta)^2 + (D_y\,\zeta)^2$, $h_2^2 = (D_x\,\eta)^2 + (D_y\,\eta)^2$. Then, if ζ has neither maximum nor minimum values within T, the surface integral of $h_1 \cdot h_2 \cdot D_\zeta\left(\frac{V}{h_2}\right)$, taken all over the area enclosed by T is equal to the line integral taken around T of $V\cos\delta$, where δ is the angle between the exterior normal drawn to T at any point, and the curve of constant η drawn through the point, and where the direction in which ζ increases is taken positive.

Similarly, if proper regard be had for signs,

$$\iint h_1 \cdot h_2 \;\; D_\eta\left(\frac{V}{h_1}\right)\, ds = \int V \sin\delta \cdot ds.$$

If through any point, P, in the coördinate plane, two arcs s_1, s_2 be drawn along which ζ and η are respectively constant, $d\,s_1 = \frac{d\,\eta}{h_2}$,

$ds_2 = \frac{d\zeta}{h_1}$, and for the element of surface $\frac{d\zeta \cdot d\eta}{h_1 \cdot h_2}$ may be used. The surface integral of $h_1 \cdot h_2 \cdot D_\zeta \left(\frac{V}{h_2}\right)$ taken over the area enclosed by T is

$$\Omega = \int\int h_1 \cdot h_2 \cdot D_\zeta \left(\frac{V}{h_2}\right) d s_1 \cdot d s_2 = \int d\,\eta \int D_\zeta \left(\frac{V}{h_2}\right) d\zeta.$$

Consider two curves OQ, $O'Q'$ along which η has respectively the constant values η_0 and $\eta_0 + \Delta\,\eta$; and let ζ increase in the directions OQ, $O'Q'$.

Let OQ cut T $2n$ times at the points P', P'', P''', ..., $P^{[2n]}$, where the values of h_2 are h_2', h_2'', h_2''', ..., $h_2^{[2n]}$, respectively, and the corresponding values of V, V', V'', V''', ..., $V^{[2n]}$. The curved line OQ makes with the normals drawn to T at P', P'', P''', etc., from within outwards the angles δ', δ'', δ''', etc., and the two curves OQ, $O'Q'$, cut out of T the $2n$ arcs $\Delta s'$, $\Delta s''$, $\Delta s'''$, ..., $\Delta s^{[2n]}$.

$$\Omega = \int d\,\eta \left[-\frac{V'}{h_2'} + \frac{V''}{h_2''} - \frac{V'''}{h_2'''} + \frac{V^{[\text{IV}]}}{h_2^{[\text{IV}]}} - \dots + \frac{V^{[2n]}}{h_2^{[2n]}} \right],$$

where the integration is to be extended over all values of η which occur within T.

The angles δ', δ''' .... $\delta^{[2n-1]}$, or their negatives, are all obtuse and their cosines are negative, but the angles δ'', $\delta^{[\text{IV}]}$, ... $\delta^{[2n]}$, or their negatives, are all acute and their cosines are positive, so that at every point, $P^{[k]}$, where OQ cuts T we have

$$\underset{\Delta\,\eta \doteq 0}{\text{Limit}} \left\{ \frac{(-1)^k \cdot \Delta\, s^{[k]} \cdot \cos \delta^{[k]}}{\frac{\Delta\,\eta}{{}_2^{[k]}}} \right\} = 1,$$

and in the expression for Ω we may write $(-1)^k \cdot \cos \delta^{[k]}\, d\, s^{[k]}$ for $\frac{d\,\eta}{h_2^{[k]}}$.

Hence,

$$\Omega = \int \left[V' \cos \delta'\, d s' + V'' \cos \delta''\, ds'' + \dots + V^{[2n]} \cos \delta^{[2n]}\, d\, s^{[2n]} \right],$$

where the sign of integration directs us to find a similar expression to that in the brackets for every pair of consecutive curves of constant η which cut T, and to find the limit of the sum of the whole. This is evidently equivalent to integrating $V \cos \delta$ all around the curve T.

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